Right-angled Coxeter groups and Hecke C^*-algebras

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Gongfest
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Outline

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2. Infinite reduced words (joint with Thomas Lam, 2015)

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3. Hecke algebras and Hecke C^*-algebras

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2. Infinite reduced words (joint with Thomas Lam, 2015)
3. Hecke algebras and Hecke C^*-algebras
4. Simplicity of Hecke C^*-algebras (Klisse 2023)

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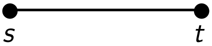
The right-angled Coxeter group (RACG) with defining graph Γ is

$$W_\Gamma = \langle S \mid s^2 = 1 \, \forall s \in S, \, st = ts \iff \{s, t\} \in E\Gamma \rangle$$

Examples of RACGs

$\Gamma_1 =$ 

$\Gamma_2 =$ 

- $W_{\Gamma_1} = \langle s, t \mid s^2 = t^2 = 1 \text{ and } st = ts \rangle \cong C_2 \times C_2$

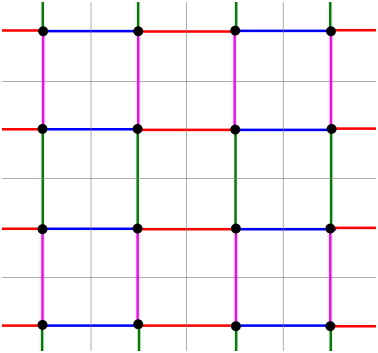
- $W_{\Gamma_2} = \langle s, t \mid s^2 = t^2 = 1 \rangle \cong D_\infty$

Examples of RACGs

If Γ a 4-cycle then

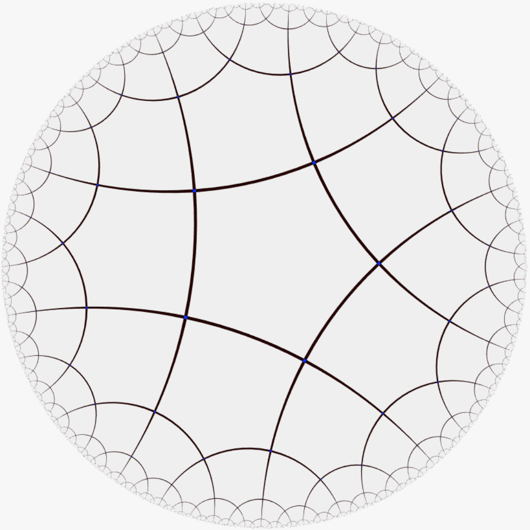
$$W_\Gamma = \langle s_1, s_2, s_3, s_4 \rangle = \langle s_1, s_3 \rangle \times \langle s_2, s_4 \rangle \cong D_\infty \times D_\infty$$

is group generated by reflections in sides of square:



Examples of RACGs

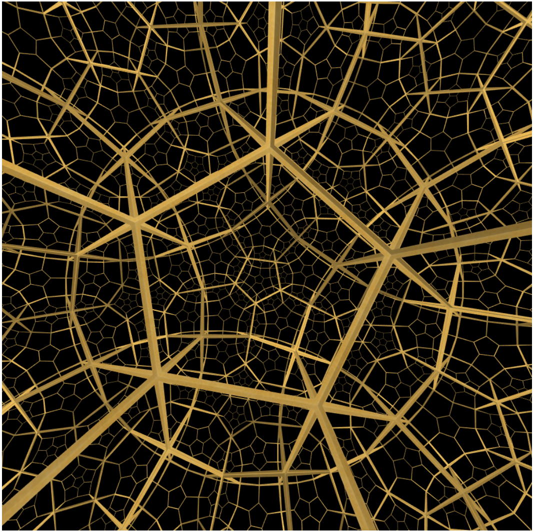
Group generated by reflections in sides of right-angled hyperbolic pentagon:



Source: Jeff Weeks' software KaleidoTile.

Examples of RACGs

Group generated by reflections in sides of right-angled hyperbolic dodecahedron:



Source: Jeff Weeks' software CurvedSpaces.

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1. *any two reduced words for w are related by a (finite) sequence of shuffles $s_i s_j \leftrightarrow s_j s_i$, where $\{s_i, s_j\} \in E\Gamma$.*
2. *a word $w = s_1 \cdots s_n$ is reduced $\iff$ it cannot be shortened by a (finite) sequence of shuffles and deletions of $s_i s_i$.*

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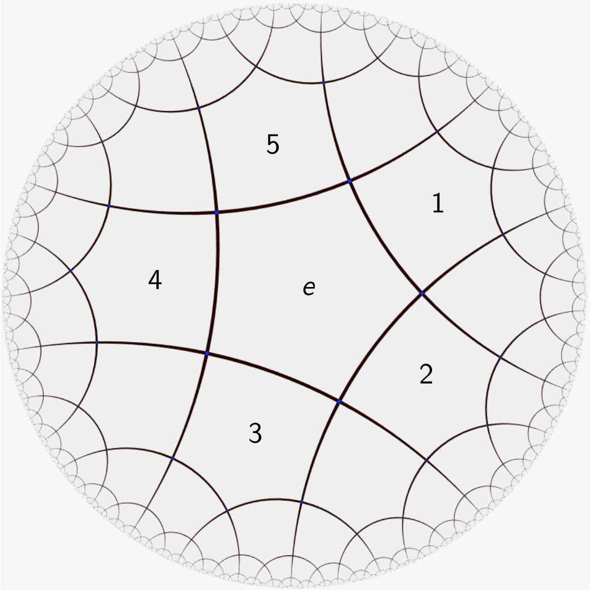
Example

If $W_\Gamma = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1 \rangle \cong D_\infty$, the only two infinite reduced words are:

$$(1, 2, 1, 2, 1, \ldots) = (1, 2)^\infty \quad \text{and} \quad (2, 1, 2, 1, 2, \ldots) = (2, 1)^\infty.$$

Example of infinite reduced words

Elements of $W_\Gamma = \langle s_1, \ldots, s_5 \rangle \longleftrightarrow$ pentagons.



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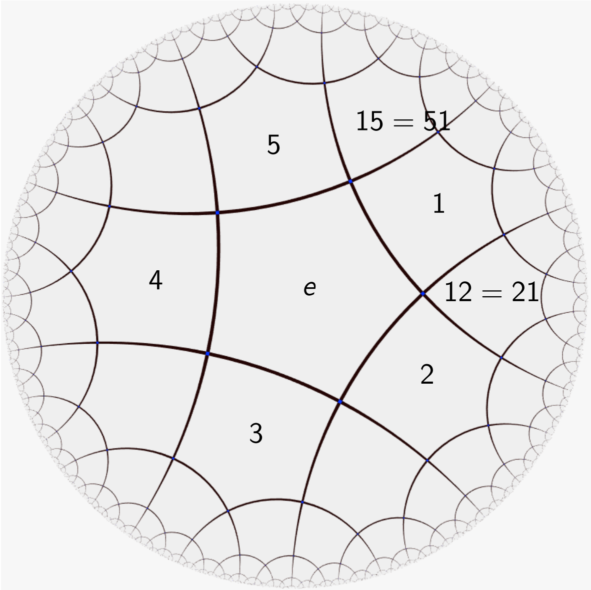

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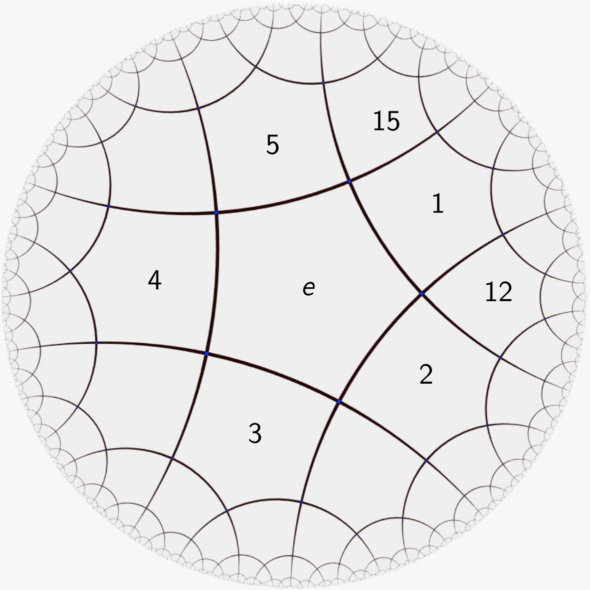

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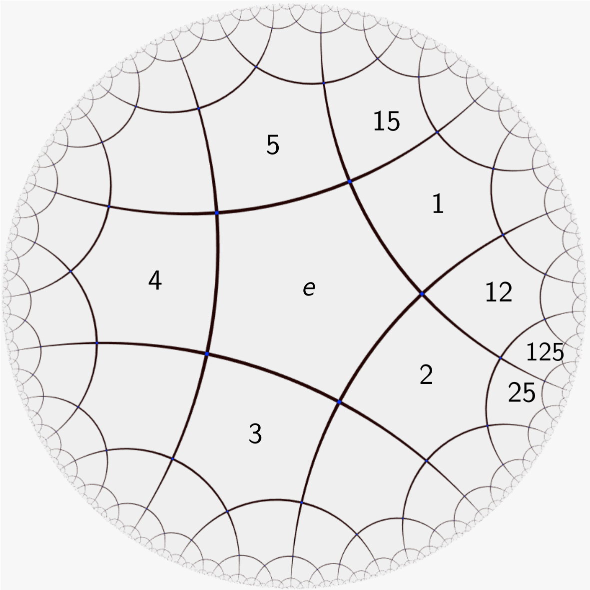

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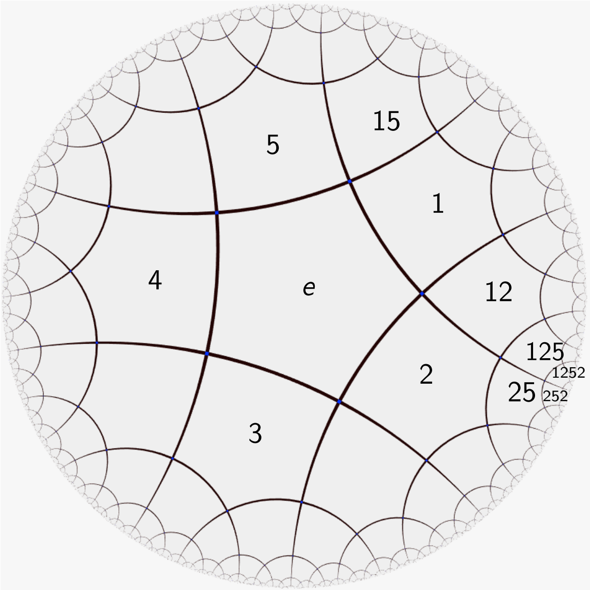

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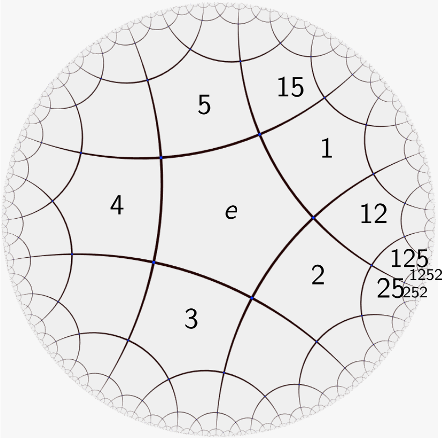

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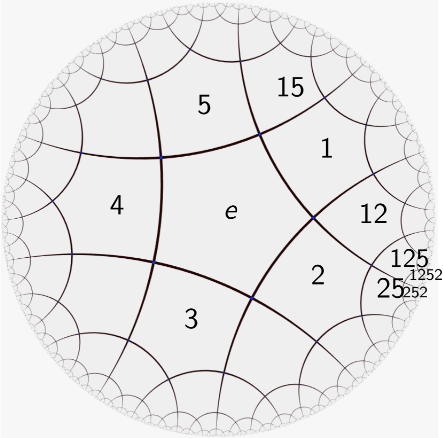

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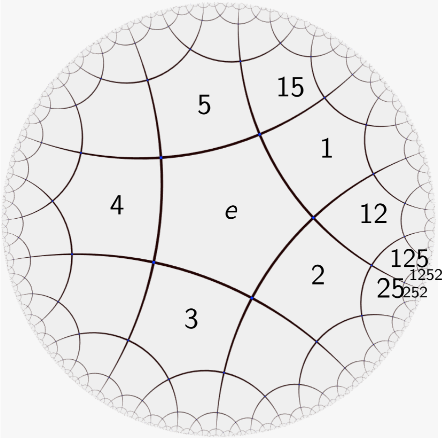

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but there's no way to shuffle $(25)^\infty \to 1(25)^\infty$.

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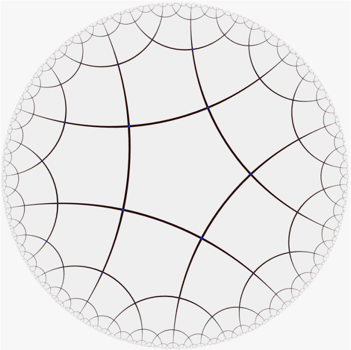
T–Lam (2015) related this partial order to the topology of the boundary of W_Γ.



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Exercise: T_s invertible with

$$T_s^{-1} = (q^{-1} - 1) T_1 + q^{-1} T_s$$

so all T_w invertible.

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The (reduced) Hecke C^*-algebra $C^*_{\Gamma,q}$ of W_Γ is the norm-closure of $C_{\Gamma,q}$.

Simplicity of Hecke C^*-algebras

W_Γ has Poincaré series

$$W_\Gamma(z) = \sum_{w \in W} z^{\ell(w)} = \sum_{k=0}^{\infty} c_k z^k$$

where $c_k = \#\{w \in W_\Gamma \mid \ell(w) = k\}$.

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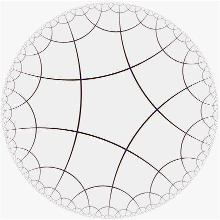
If $W_\Gamma = \langle s_1, s_2, s_3, s_4, s_5 \rangle$ then

$$W_\Gamma(z) = 1 + 5z + 15z^2 + 40z^3 + 105z^4 + \cdots$$



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Define

$$R_\Gamma^{\pm 1} = \{q, q^{-1} \mid q > 0 \text{ and } W_\Gamma(q) \text{ converges}\}$$

and write $\overline{R_\Gamma^{\pm 1}}$ for its closure in $\mathbb{R}_{>0}$.

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so $R_\Gamma^{\pm 1} = \mathbb{R}_{>0} \setminus \{1\}$ and $\overline{R_\Gamma^{\pm 1}} = \mathbb{R}_{>0}$.

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Radius of convergence is $\frac{3-\sqrt{5}}{2} \sim 0.38$ so

$$\overline{R_\Gamma^{\pm 1}} = \left(0, \frac{3-\sqrt{5}}{2}\right] \cup \left[\frac{2}{3-\sqrt{5}}, \infty\right)$$

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Proof.

Topology of the boundary of W_Γ. □