Scattering theory and operator algebras: a fruitful exchange

Serge Richard

Nagoya University

Gongfest 2025

June 26

Scattering theory: a comparison theory

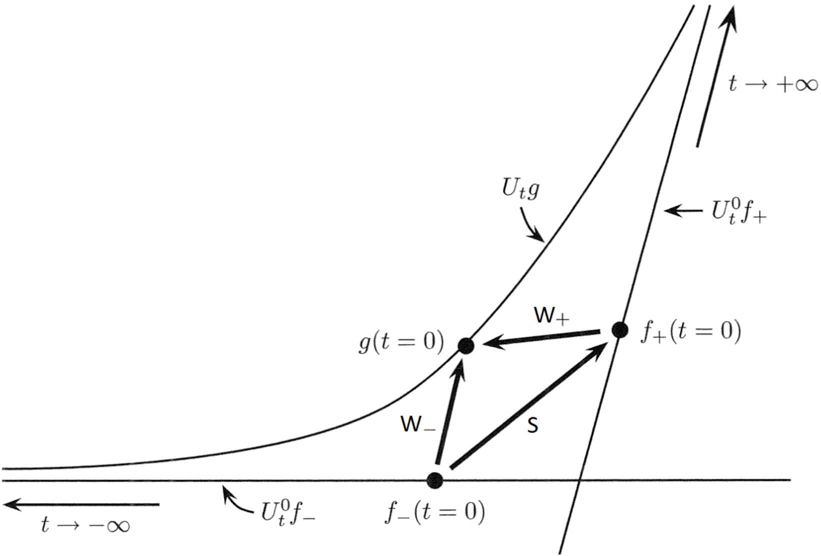

Figure: Representation of scattering theory

Scattering theory (self-adjoint theory)

$\mathcal{H}$ a Hilbert space, H_0, H self-adjoint operators in $\mathcal{H}$ with

- H_0 purely absolutely continuous,
- $W_\pm := s - \lim_{t\to\pm\infty} \mathrm{e}^{itH}\mathrm{e}^{-itH_0} E_{\mathrm{ac}}(H_0)$ exist,
- $\mathrm{Ran}(W_-) = \mathrm{Ran}(W_+) = \big(1 - E_{\mathrm{p}}(H)\big)\mathcal{H}$.

Then,

- The wave operators $W_\pm$ are isometries with
$$W_\pm^* W_\pm = 1 \quad \text{and} \quad W_\pm W_\pm^* = 1 - E_{\mathrm{p}}(H),$$
- The scattering operator $S := W_+^* W_-$ is unitary.

Index map in K-theory [RLL]

Let

$$0 \longrightarrow \mathcal{J} \hookrightarrow \mathcal{E} \xrightarrow{q} \mathcal{Q} \longrightarrow 0$$

be a short sequence of C^*-algebras, with $\mathcal{E}$ unital. Let U be a unitary element in $\mathcal{Q}$ which has a lift to a partial isometry W in $\mathcal{E}$. Then $1 - W^*W$ and $1 - WW^*$ are projections in $\mathcal{J}$, and

$$\operatorname{ind}\left([U]_1\right) = [1 - W^*W]_0 - [1 - WW^*]_0.$$

If $W_- \in \mathcal{E}$ with $q(W_-)$ unitary, then

$$\operatorname{ind}\left([q(W_-)]_1\right) = -[E_{\mathrm{p}}(H)]_0.$$

But how to choose $\mathcal{E}$? Who is $q(W_-)$? Can we get an equality between numbers ?

Filling the framework: Topological Levinson's Thm

The program:

- Fix a pair (H, H_0),
- Get a rather explicit formula for W_-,
- Guess a suitable algebra $\mathcal{E}$ with $W_- \in \mathcal{E}$,
- Find $\mathcal{J}$ such that $q(W_-)$ is unitary,

We then get

$$\operatorname{ind}\big([q(W_-)]_1\big) = -[E_{\mathrm{p}}(H)]_0.$$

- Deduce a numerical equality from this relation.

This program has been successful on several examples.

¿ What can scattering theory bring back to operator algebras ?

Example I: The operators $H_{m,\kappa}$

For $|\mathrm{Re}(m)| < 1$, $m \neq 0$, and $\kappa \in \mathbb{C} \cup \{\infty\}$ we set in $L^2(\mathbb{R}_+)$

$$H_{m,\kappa} := -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{r^2}$$

with

$$\mathcal{D}(H_{m,\kappa}) := \Big\{ f \in L^2(\mathbb{R}_+) \mid H_{m,\kappa} f \in L^2(\mathbb{R}_+) \text{ and } \exists c \in \mathbb{C} \text{ s.t.} \\ f(r) \sim c\big(\kappa r^{\frac{1}{2}-m} + r^{\frac{1}{2}+m}\big) \text{ near } 0 \Big\}.$$

$H_{m,\kappa}$ is self-adjoint if (m, κ) belongs to

$$\Omega_{\mathrm{sa}} := \{(m,\kappa) \mid m \in (-1,1) \setminus \{0\} \text{ and } \kappa \in \mathbb{R}, \\ \text{or } m \in i\mathbb{R} \setminus \{0\} \text{ and } |\kappa| = 1\},$$

and $\sharp\sigma_{\mathrm{p}}(H_{m,\kappa}) = \dim\big(E_p(H_{m,\kappa})\big) \in \mathbb{N} \cup \{\infty\}$.

Example I: The wave operators

$W_-(H_{m,\kappa}, H_{m',\kappa'})$ are partial isometries and unitarily equivalent in $L^2(\mathbb{R})$ to

$$\boxed{\begin{aligned}&\mathcal{W}_-(H_{m,\kappa}, H_{m',\kappa'})\\ &= \Xi_{\frac{1}{2}}(-D)\big(\Xi_m(D) - \varsigma\Xi_{-m}(D)\mathrm{e}^{2mX}\big)\frac{\mathrm{e}^{-i\frac{\pi}{2}m}}{1-\varsigma\mathrm{e}^{-i\pi m}\mathrm{e}^{2mX}}\\ &\quad\times\frac{\mathrm{e}^{+i\frac{\pi}{2}m'}}{1-\varsigma'\mathrm{e}^{+i\pi m'}\mathrm{e}^{2m'X}}\big(\Xi_{m'}(-D) - \varsigma'\mathrm{e}^{2m'X}\Xi_{-m'}(-D)\big)\Xi_{\frac{1}{2}}(D).\end{aligned}}$$

with $\varsigma := \kappa\frac{\Gamma(-m)}{\Gamma(m)}$, $\quad \varsigma' := \kappa'\frac{\Gamma(-m')}{\Gamma(m')}$ and

$$\mathbb{R} \ni \xi \mapsto \Xi_m(\xi) := \mathrm{e}^{i\ln(2)\xi}\,\frac{\Gamma(\frac{m+1+i\xi}{2})}{\Gamma(\frac{m+1-i\xi}{2})} \in \mathbb{C}\ .$$

Example I: Classification

$\mathcal{W}_-(H_{m,\kappa}, H_{m',\kappa'})(x,\xi)$ is a ΨDO of order 0 in ξ, and...

The Fredholm case

- The coef. have limits at $x = \pm\infty$,
- Already existing framework, classical situation, studied in [NPR].

The semi-Fredholm case (I)

- The coef. are periodic,
- Atiyah's L^2-index theorem [Ati].

The semi-Fredholm case (II)

- The coef. are asymptotically periodic,
- C^*-algebras studied in [CM],
- 2-link ideal chain and two different index theorems.

The almost periodic case

- The coef. are almost periodic,
- C^*-algebras and index theorems in [CDSS, CMS, Shu].

Example I: The Fredholm case

Let $m \in (-1,1) \setminus \{0\}$, $\kappa \in \mathbb{R}$, and $(m', \kappa') = \left(\frac{1}{2}, 0\right)$. Then,

$$\mathcal{W}_-(H_{m,\kappa}, H_{\frac{1}{2},0}) \in \mathcal{E}_\Box := C^*\Big(a(X)b(D) \mid a, b \in C\big([-\infty,\infty]\big)\Big).$$

- $\mathcal{E}_\Box / \mathcal{K}\big(L^2(\mathbb{R})\big) \cong C(\Box) \cong C(\mathbb{S}^1)$
- The quotient map q is given by

$$a(X)b(D) \mapsto \Big(b(-\infty)\,a, a(+\infty)\,b, b(+\infty)\,a, a(-\infty)\,b\Big)$$

- $\Gamma_{m,\kappa;\frac{1}{2},0} := q\big(\mathcal{W}_-(H_{m,\kappa}, H_{\frac{1}{2},0})\big)$ has 4 components, one of them is unitarily equivalent to the scattering operator.

Theorem (Fredholm case)

$$\mathrm{Wind}_\Box(\Gamma_{m,\kappa;\frac{1}{2},0}) = -\,\mathrm{Index}\,\big(\mathcal{W}_-(H_{m,\kappa}, H_{\frac{1}{2},0})\big) = \#\sigma_{\mathrm{p}}(H_{m,\kappa}).$$

Example II: Unitary theory (discrete time)

In $\mathcal{H} := \ell^2(\mathbb{Z} \times \mathbb{N})$, let U_0, U_1 be unitary operators,

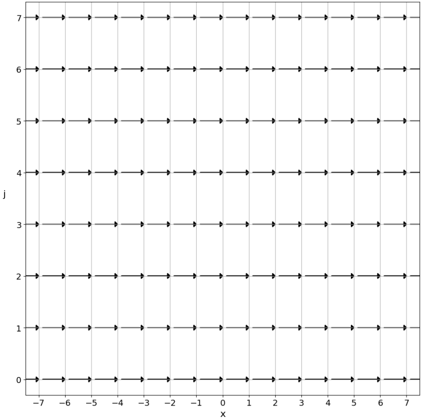

Figure: Action of U_0

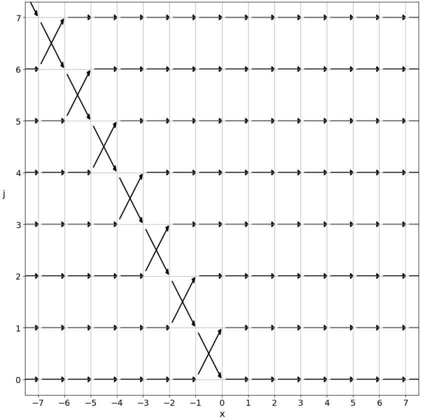

Figure: Action of U_1

and the wave operators $W_\pm(U_1, U_0) := s - \lim_{n\to\pm\infty} U_1^{-n} U_0^n$.

Example II: Wave operators $W_\pm(U_1, U_0)$

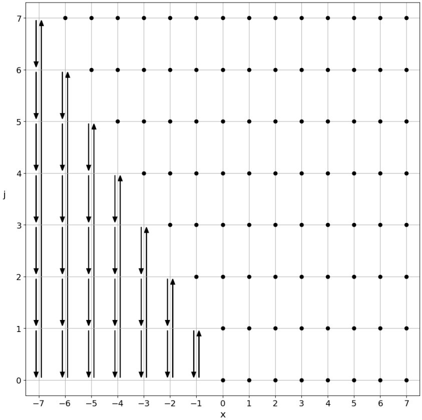

Figure: Action of $W_+(U_1, U_0)$

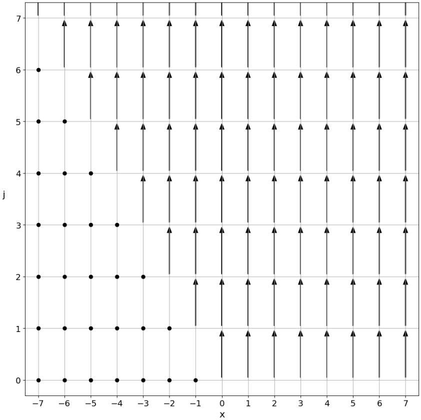

Figure: Action of $W_-(U_1, U_0)$

Observe that $\mathrm{Ran}\left(W_+(U_1, U_0)\right) = \mathcal{H}$ but $\mathrm{Ran}\left(W_-(U_1, U_0)\right) \neq \mathcal{H}$.

Example II: U_2 with one invariant point

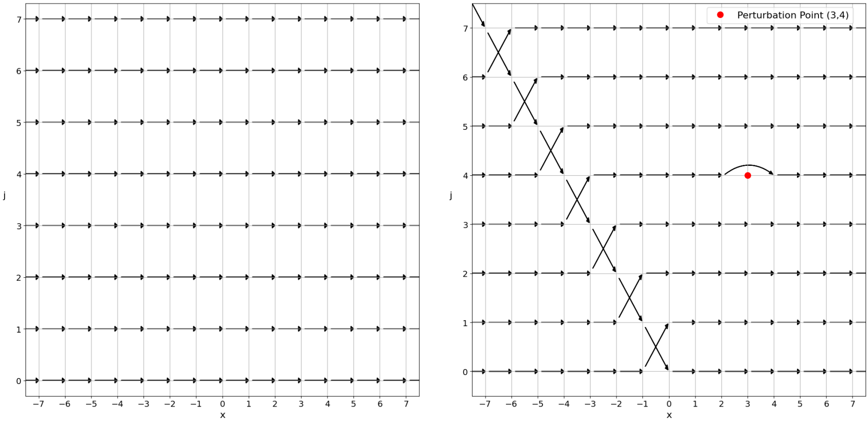

Figure: Action of U_0 Figure: Action of U_2

and the wave operators $W_{\pm}(U_2, U_0) := s - \lim_{n\to\pm\infty} U_2^{-n} U_0^{n}$.

Example II: Wave operators $W_{\pm}(U_2, U_0)$

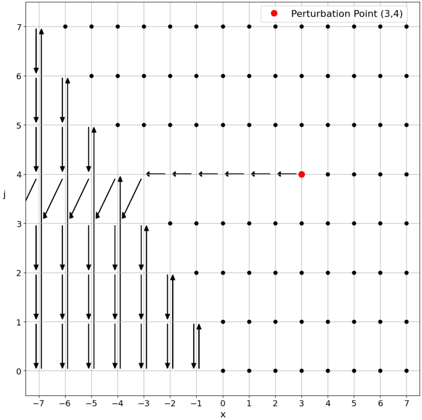

Figure: Action of $W_{+}(U_2, U_0)$

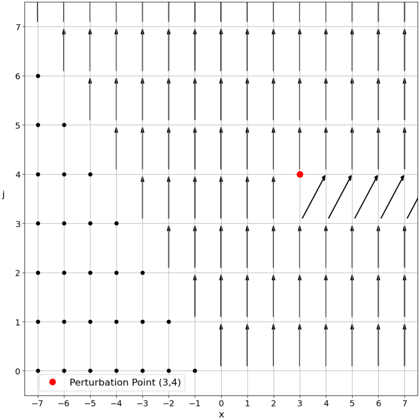

Figure: Action of $W_{-}(U_2, U_0)$

¿ All these operators are isometries, what do they illustrate ?

Example II: Wold's decomposition

On $\ell^2(\mathbb{N})$ with the standard basis $\{\delta_n\}_{n\in\mathbb{N}}$, set $\mathcal{S}\delta_n = \delta_{n+1}$.

Theorem (Wold's decomposition)

If W is an isometry on a Hilbert space $\mathcal{H}$, then there is a cardinal number α and a unitary operator V (possibly vacuous) such that W is unitarily equivalent to $\mathcal{S}^{(\alpha)} \oplus V$.

These examples provide a perfect visualization of this theorem. Other general properties of scattering theory can be illustrated with these operators.

Conclusion

- Operator algebras provides a natural framework for new investigations on scattering theory
- In return scattering theory provides non-trivial examples
- What is the next step ?

THANK YOU

References

[Ati] M.F. Atiyah, *Elliptic operators, discrete groups and von Neumann algebras*, Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974), pp. 43–72, Astérisque **32-33**, Soc. Math. France, Paris, 1976.

[CDSS] L.A. Coburn, R.G., Douglas, D.G. Schaeffer, I.M. Singer, *C^*-algebras of operators on a half-space II : index theory*, Inst. Hautes Etudes Sci. Publ. Math. **40**, 69–79, 1971.

[CMS] L.A. Coburn, R.D. Moyer, I.M. Singer, *C^*-algebras of almost periodic pseudo-differential operators*, Acta Math. **130**, 279–307, 1979.

[CM] H.O. Cordes, S.T. Melo, *An algebra of singular integral operators with kernels of bounded oscillation, and application to periodic differential operators*, J. Differential Equations **75** no. 2, 216–238, 1988.

[FR] R. Firdaus, S. Richard, *Discrete time scattering and Wold's decomposition in pictures*, submitted, 2025.

[IR1] H. Inoue, S. Richard, *Index theorems for Fredholm, semi-Fredholm and almost periodic operators: all in one example*, J. Noncommut. Geom. **13**, 1359–1380, 2019.

[NPR] F. Nicoleau, D. Parra and S. Richard, *Does Levinson's theorem count complex eigenvalues?*, J. Math. Phys. **58**, 102101, 7pp., 2017.

[RLL] M. Rørdam, F. Larsen, N. Laustsen, *An introduction to K-theory for C^*-algebras*, London Mathematical Society Student Texts **49**, Cambridge University Press, Cambridge, 2000.

[Shu] M.A. Shubin, *The spectral theory and the index of elliptic operators with almost periodic coefficients*, Russian Math. Surveys, **34** no. 2, 109–158, 1979.