Index theory in the Gong

Adam Rennie
University of Wollongong, NSW, Australia
The Gong

June 25, 2025

Bringing index theory to the Gong

I will chart my time and index theory at the Gong thru Gongfests.

This is an imperfect and sporadic record, but probably better than my memory. No one gets two Gongfest mentions...

I came to the Gong in 2012 with three freshly started PhD students:

Chris Bourne: bulk-edge correspondence for topological insulators

Iain Forsyth: equivariant spectral triples and noncommutative manifolds-with-boundary

Koen van den Dungen: Lorentzian noncommutative geometry

Arrival

The first Gongfest in 2012 was held to introduce my students and I to the folk at the Gong, and conversely.

At that time of our arrival the Gong folk were Aidan, Jacqui Ramagge, David Pask and Nathan Brownlowe (according to the Gongfest schedule), but Mike Whittaker and Sam Webster were also around then.

As before, and for many Gongfests since, Alan Carey was a speaker and also an important part of the community.

My first Gong honours student: Mark Bugden, quantum statistical mechanics

The baggage I brought

I had been involved in developing index theory for KMS states of periodic actions. All the natural examples were one-dimensional. My first student Roger Senior and I went looking for higher dimensional examples in quantum group land.

The tool that imposed itself on us was the Kasparov product. Here the Hopf fibration $SU(2) = S^3 \to S^2$ with fibres the circle can be q-deformed to become $SU_q(2) \to S^2_q$.

The geometry of S^2_q was well-known, and there is an easy way to build a Kasparov module for the fibration. Banging them together seemed straightforward. Sigh.

The baggage I brought

So my bromance with Bram Mesland began in 2011, and my work on the constructive Kasparov product. Given C^*-algebras A, B, C we have a $\mathbb{Z}$-bilinear, associative product with is functorial in all imaginable ways

$$KK(A, B) \times KK(B, C) \to KK(A, C).$$

On a good day...

$$(\mathcal{A}, X_B, S) \times (\mathcal{B}, Y, T) = (\mathcal{A}, X \otimes_B Y, S \otimes 1 + 1 \otimes_\nabla T)$$

provides an explicit formula for computing the product. It is known to fail, particularly in dynamical settings, and of course $SU_q(2)$.

That kept me busy until the mid-teens.

I had also had an idea about boundaries, as in manifolds-with-boundary. My madness around this led to Chris studying the bulk-boundary correspondence for topological insulators and Iain studying manifolds-with-boundary.

Chris' work turned out to have nothing to do with my idea for boundaries but lots to do with KK. Iain thoroughly debunked my idea for manifolds-with-boundary, proving much better results, and also doing some equivariant KK. Great!

Koen came from a physics background, and so we looked at Lorentzian geometry in noncommutative frameworks, and wound up deep in Kasparov theory again.

Extensions

Roger:

$$C(S_q^2) \leftrightarrow SU_q(2)$$

Chris:

$$0 \to C(S^1) \otimes \mathbb{K} \to T \to A_\theta \to 0$$

Iain:

$$0 \to C_0(M) \to C(\overline{M}) \to C(\partial M) \to 0$$

Koen:

$$ST + TS \text{ small} \Rightarrow S + T \text{ self-adjoint.}$$

Using the extra geometric information here to build Kasparov classes that we could usefully take products with turned out to be the key.

Settling in

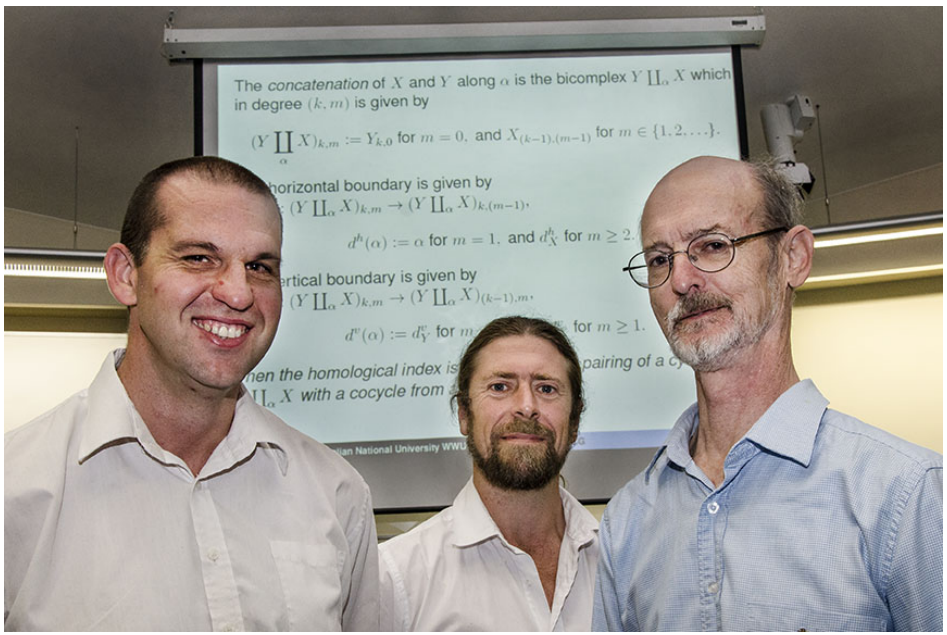
Gongfest II: Fedor's Revenge, 2013... Bram Mesland, Dave Robertson, Adam Sierakowski, someone broke an arm.

Gongfest III: Beyond ThunderGong, 2014... Efren Ruiz, Peter Hochs, Bruno Iochum, Nate Brown breaks a leg

Settling in

Gongfest II: Fedor's Revenge, 2013... Bram Mesland, Dave Robertson, Adam Sierakowski, someone broke an arm.

Gongfest III: Beyond ThunderGong, 2014... Efren Ruiz, Peter Hochs, Bruno Iochum, Nate Brown breaks a leg



Settling in

Gongfest IV: Donkey Gong (Andreas), 2015... Mitch Hawkins, James Rout, Andreas Anderssen, Marco Matassa, James Fletcher

During this period there were new PhD students:

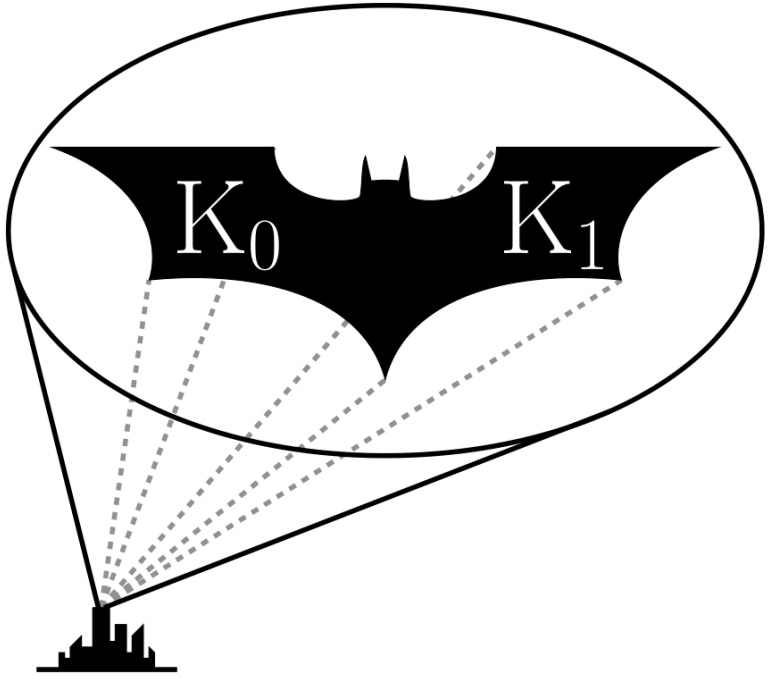
Andreas Andersson, Nonunital spectral flow

James Fletcher, Product systems from iterating the Cuntz-Pimsner construction, with Aidan

and more honours students: Alex Gerhardt-Bourke (Poincaré duality, with Aidan), Alex Mundey (Bott periodicity), Lachy McDonald (Atiyah-Singer index theorem).

Matrix

Gongfest V: Enter the Matrix 2016.



Manifolds-with-boundary

Iain and I, with Bram and Magnus Goffeng, showed that the long-standing folk-theorems about how noncommutative manifolds-with-boundary should work were both not wrong and also woefully naive.

For Riemannian manifolds-with-boundary, the Dirac operator with Dirichlet boundary conditions is symmetric but not self-adjoint. It is still nice enough to define a (relative) spectral triple.

Generalising the idea of mfld-w-bdry to the noncommutative world using this model fails almost every single subsequent test, unless many other assumptions are satisfied.

Pseudo-Riemannian manifolds and manifolds-with-boundary

Koen and I formulated a new way of incorporating Lorentzian manifolds into noncommutative geometry, but it relied on technical tools for the Kasparov product that just didn't do the job. We pestered the experts, and ultimately Bram and Matthias Lesch fixed these issues and the method worked perfectly.

The Dirac operator on a pseudo-Riemannian manifold decomposes as

$$D = D_T + D_S, \quad D_T^* \supset -D_T, \quad D_S^* \supset D_S$$

so that $iD_T + D_S$ is symmetric. Even if iD_T, D_S are both self-adjoint, more was needed to prove that their sum is self-adjoint, and this was the technical issue solved by Bram and Matthias.

Topological insulators

Chris, Alan and I (and later Johannes Kellendonk) showed that the bulk-edge correspondence for topological insulators followed from associativity of the Kasparov product. Even more, the relation could be expressed directly in terms of physical variables.

This built on an idea from Johannes, who showed that the algebras of observables for a topological-insulator-with-bdry could be factored as

$$0 \to O_{edge} \otimes \mathbb{K} \to T \to O_{bulk} \to 0.$$

By representing this extension as an explicit cycle with class $[ext] \in KK^1(O_{bulk}, O_{edge})$, we could relate the bulk conductivity (pairing of the bulk with Fermi projector P_F) to the boundary current (pairing of the edge with the unitary $P_F \otimes ext$) via associativity of the Kasparov product.

Can add discrete symmetries and antisymmetries to accommodate all kinds of topological insulators.

Sweet sixteen

At this time Bram and I completed our work on the Kasparov product, surprising ourselves with a new proof of existence of the product.

Aidan, Dave Robertson, Bram, Magnus and I built on an idea of Bram and Magnus to study Cuntz-Pimsner algebras. These dynamical algebras generalise Cuntz-Krieger algebras, and B-M had completely unpacked their noncommutative geometry. The same ideas extended to $*$some$*$ Cuntz-Pimsner algebras.

It also led to my first work with Francesca Arici, the better half of Bramcesca.

The Cuntz-Pimsner extension

Given a self-correspondence (A, E_A), one builds the Fock module $F_E = \oplus_{n\geq 0} E^{\otimes n}$ and the Toeplitz-Pimsner algebra T_E of creation and annihilation operators on F_E. The Cuntz-Pimsner algebra O_E is then defined (caveats aplenty) by

$$0 \to End^0_A(F_E) \to T_E \to O_E \to 0.$$

Representing the class of this defining extension $[ext] \in KK^1(O_E, A)$ in sufficiently precise ways enabled us to transfer geometric cycles for A to geometric cycles for O_E, and study when Poincaré duality for A could be transferred to O_E.

Before the storm

Gongfest VI: Making the Gong great again! (Tim) 2018 Alex Kumjian, Rhys McDonald, Nick Seaton, Michael Mampusti

Matthias Lesch spent sabbatical here in 2019 and worked with Chris, Alan and I on real spectral flow.

More new PhD students:

Lachlan MacDonald: Godbillon-Vey invariant for foliated manifolds

Alex Mundey: Topology and dynamics of fractals

and more new honours students: Jacob Bradd (Local Atiyah-Singer a la Getzler), Tess O'Brien (Quantum entanglement)

Foliations

A foliated manifold (M, F) gives rise to a (holonomy) groupoid G_F and associated C^*-algebra $C^*(G_F)$.

There is a natural and relatively easy index theory for the Dirac operator along leaves of the foliation (the so-called longitudinal index). The transverse geometry is way trickier, as most geometric objects in the transverse direction (metric, volume form) are not invariant under flowing along leaves.

The Godbillon-Vey class is an equivariant cohomology class measuring the failure of invariance of the transverse geometry. Lachy and I showed the Godbillon-Vey class could be expressed using equivariant KK-pairings, using ideas from Connes and Moscovici. The best results were in codimension 1, with higher codimension much more difficult (as usual).

Fractals

Long ago I had started studying fractals as a possible source of higher dimensional modular index problems. This is (still) not mad.

The fractals coming from iterated function systems are kinda inimical to the topological tools of C^*-algebra theory. Alex Mundey and I explored all the ways that this statement rears its ugly head: groupoids, quivers, semigroups and on and on.

The method that yielded novel invariants was a clever modification of the Cuntz-Pimsner construction. Setting up the dynamics of these spaces naively (Watatani et al) only allows one to see topological features arising from one kind of critical point (branch points). We refined the structure of the module to see more detail.

More Lorentzian stuff

For a few years in the late teens, Ben Whale started working here, commuting from Hobart!!

Ben and I had worked on time functions in Lorentzian manifolds when I was at ANU, and we restarted our work in this area again. It goes slowly due to distractions aplenty.

Whatever the distractions, Ben is an amazing Lorentzian geometer, and I will be his student for many years to come.

The dark ages

Then everything looked like shit for a bit.

Nevertheless, life went on and new work was done. At this time, I was working with Magnus a bit, and Bram a lot. One idea each from Bram and I collided, and I spent a very long time learning a lot about curvature.

Astonishingly, more new PhD students:

Angus Alexander: Topological Levinson's theorem via index pairings and spectral flow.

Ada Masters: Groups and their actions in unbounded Kasparov theory

Scattering

In scattering theory, one throws elctrons, say, at a molecule, again say, and asks what one can deduce about the molcule from how the electrons scatter off.

Astonishingly, the number of bound states the molecule can hold can be determined by the scattering data. This is called Levinson's theorem (1949).

The theorem comes with many technical caveats, all completely surmountable except for one which occurs in low dimensions, called resonant states. The task of understanding these has been inflicted upon Angus, along with many others.

Group actions

Groups act on every kind of mathematical object, and so equivariant versions of all kinds of mathematics exist. For Kasparov theory, the great G.G. wrote the rules and all was well. Until you examine geometric or physical representatives of Kasparov classes.

Following Kucerovsky (almost only, and very briefly), Ada rebuilt the representing geometric cycles of Kasparov theory to accommodate group equivariance. Her reconfiguration captures the conformal actions encoded by G.G., but there is still room to grow and accommodate more general actions.

Hyperbolicity here we come!

The enlightenment?

The need to re-meet after too long online, reprising the purpose of the first Gongfest.

Gongfest VII: Gongfestivus for the rest of us. . . Rodrigo Frausino, Jamie Gabe, Abraham Ng, Anna Duwenig, Joel Anderson, Ada Masters, Angus Alexander (2022)

The enlightenment?

The need to re-meet after too long online, reprising the purpose of the first Gongfest.

Gongfest VII: Gongfestivus for the rest of us. . . Rodrigo Frausino, Jamie Gabe, Abraham Ng, Anna Duwenig, Joel Anderson, Ada Masters, Angus Alexander (2022)

The visitors began to return. . .

Escape!

. . . and we could start to travel again. I took sabbatical in Leiden and completed the foundations of the curvature work Bram and I had begun long before.

Rebuilding

. . . and the visitors kept coming

And yet more new honours students: Kai Buckman (Tomita-Takesaki), Alex Paviour (Spin-Statistics theorem)

And a first Masters student: Alex Paviour: Action functionals in noncommutative particle physics.

And so here we are...

And so here we are...

Gongfest VIII:

So long and thanks for all the fish