SELF-SIMILAR GROUPS AT GAGTA

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Gongfest: so long, and thanks for all the fish
26 June 2025

Slides at `sites.google.com/site/melderau/talks`

ABOUT

GAGTA (geometric and asymptotic group theory with applications) is an annual conference running since 2006.

Themes vary each year, sometimes it can be quite geometric or analytic, but other times algorithmic,

and *applications* used to stand for non-abelian group based cryptography, but this year stood for AI and Lean proofs.

There were *a lot* of talks, and the following were specifically about self-similar groups.

1. Tatiana Nagnibeda: Generalized word problem in automata groups[1]
2. Laurent Bartholdi: The domino problem on groups and Schreier graphs[2]
3. Slava Grigorchuk: The Collatz $3x+1$ conjecture and self-similar groups
4. Dima Savchuk: Diagonal actions of groups acting on rooted trees
5. Ben Steinberg: The homology of groupoids and groups associated to self-similar groups[3]

[1] Bishop, D'Angeli, Matucci, Nagnibeda, Perego, and Rodaro, *On the subgroup membership problem in bounded automata groups*, 2024.
[2] Bartholdi, "Monadic second-order logic and the domino problem on self-similar graphs", 2022.
[3] Miller and Steinberg, *Homology and K-theory for self-similar actions of groups and groupoids*, 2024.

ABOUT CONTINUED

My talk will be a quick review of what is a self-similar group,

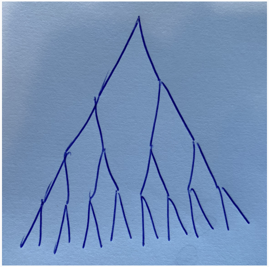
then I will focus on the talks of Grigorchuk and Nagnibeda,

since these had connections to formal language theory which I like.

The talks were recorded and are available at
`https://web.stevens.edu/algebraic/Stevens2025/program.php`

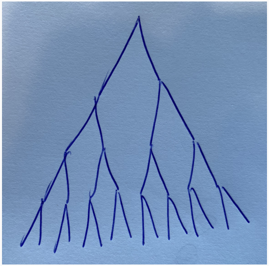
WHAT IS A SELF-SIMILAR GROUP: QUICK VERSION

Let T_d be the d-ary rooted tree, and $\mathrm{Aut}(T_d)$ the group of graph isomorphisms from T_d to T_d (so they must fix the root, and hence each level).



An element $g \in \mathrm{Aut}(T_d)$ can perform some permutation $\sigma \in S^d$ of the level 1 nodes, then for each node u in level 1 act as some other automorphism of a tree with root u.

We use the notation $g|_u$ for this automorphism, called the *restriction* of g to u.

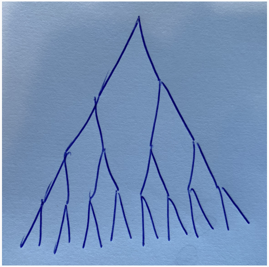


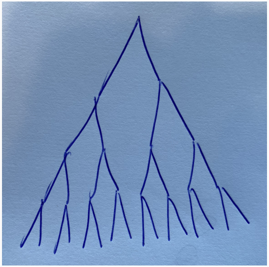
Every element can be encoded by $\sigma(g|_0, \ldots, g|_{d-1})$.

A subgroup $G \leq \mathrm{Aut}(T_d)$ is *self-similar* if $g|_u \in G$ for every $g \in G$ and every node $u \in T_d$[4].

[4] Nekrashevych, *Self-similar groups*, 2005.

Let $S_2 = \{e, \sigma\}$, and $p = \sigma(q,q)$, $q = e(q,p)$ be two automorphisms of T_2.





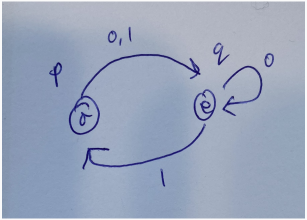
Then $\langle p, q \rangle \cong D_\infty$[5].

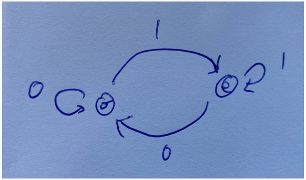
[5] Ortega and Sanchez, “The homology of the groupoid of the self-similar infinite dihedral group”, 2022.

EXAMPLES

$p = \sigma(q,q)$, $q = e(q,p)$

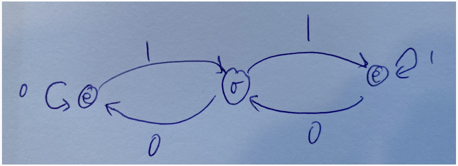
In this case we can describe p, q using the following automaton.



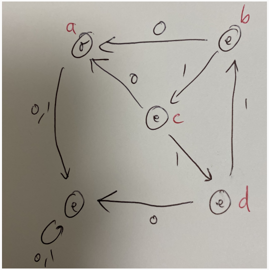


The two states of the automaton on the right generate a different group: the lamplighter group $\mathbb{Z} \wr C_2$

The Baumslag-Solitar group BS(1,3) is isomorphic to the group generated by the states of



The (first) Grigorchuk group is generated by the states of



If a self-similar group is generated by the states of some (not necessarily finite) automaton, it is called an *automata group*

of type (d, n) if it a subgroup of $\mathrm{Aut}(T_d)$ and the automaton has n states.

Grigorchuk discussed the problem of classifying the automaton groups of type (d, n) for small values of d, n

and suggested it for "the younger generation" to work on[6,7].

[6] Bondarenko, Grigorchuk, Kravchenko, Muntyan, Nekrashevych, Savchuk, and Šunić, "On classification of groups generated by 3-state automata over a 2-letter alphabet", 2008.

[7] Davis, Elder, and Reeves, "Non-contracting groups generated by $(3, 2)$-automata", 2014.

Define $T\colon \mathbb{N} \to \mathbb{N}$ by $T(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{3n+1}{2} & n \text{ odd} \end{cases}$.

For $n \leq 2^{68}$ iterating this map arrives at the orbit $1 \leftrightarrow 2$. Conjecture: same for all n.

Grigorchuk linked it with a self-similar group as follows.

Extend to ring of dyadic integers $\mathbb{Z}_2$:

$$\begin{cases} T(2z) = z \\ T(1+2z) = \mu_2(z) \end{cases}$$

where $\mu_2(z) = 3z+2$.

$\mathbb{Z}_2$ is in bijection with right-infinite strings $\{0,1\}^\infty$ (the leaves of T_2), where the Collatz map becomes:

$$\begin{cases} T(0w) = w & \text{(divide "even" by 2)} \\ T(1w) = \mu_2(w) \end{cases}$$

He then introduced an automorphism of T_2 called the *Terras map* which he called the "hero" of his talk.

$$\gamma(w) = y_0 y_1 \ldots$$

where y_i is the parity of $T^i(w)$. See[8].

[8] Terras, "A stopping time problem on the positive integers", 1976.

Note $\mathbb{Z}^+$ are strings ending in 0^∞ and $\frac{1}{3}\mathbb{Z}$ are infinite strings that eventually periodic with periods 0 and 01.

Conjecture (Collatz in terms of γ)

$$\gamma(\mathbb{Z}^+) \subseteq \frac{1}{3}\mathbb{Z}$$

Other observations:

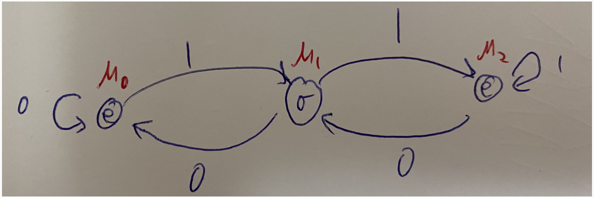
- γ conjugates T to the one-sided shift map (delete the first letter) ("the most important map in dynamics")
- Intersect with $\mathbb{Q}$:

Conjecture (Periodicity conjecture of Lagarias 1985)

$$\gamma(\mathbb{Z}_2 \cap \mathbb{Q}) = \mathbb{Z}_2 \cap \mathbb{Q}$$

Definition

Define the group $\mathscr{C}$ generated by γ and the 3 states generating BS(1,3) the *Collatz group*.



Definition

Define the group $\mathscr{C}$ generated by γ and the 3 states generating $\mathrm{BS}(1,3)$ the *Collatz group*.

This group is (clearly) finitely generated, and is self-similar, but the automaton generating it is *infinite*.

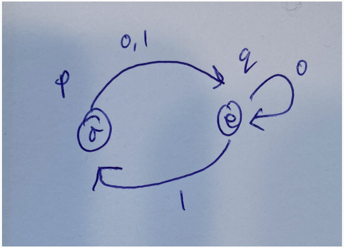
Questions:

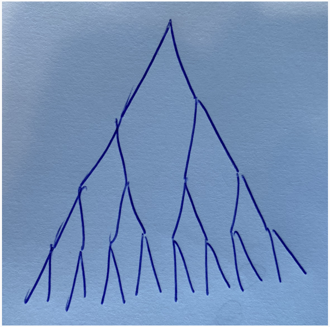
- is $\mathscr{C}$ isomorphic to the free product of $\langle\gamma\rangle$ and $\mathrm{BS}(1,3)$ (the group generated by the other states)?
- Are the Schreier graphs of the level n stabilizers *expander graphs*?

A node u of T_d is *active* for $g \in G$ if $g|_u = \sigma(g|_{u1}, \ldots, g|_{ud})$ with $\sigma \neq e$.

The *portrait* of $g \in G$ is (a picture of) the set of all active nodes for g.

Eg:





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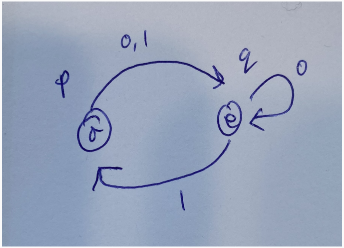
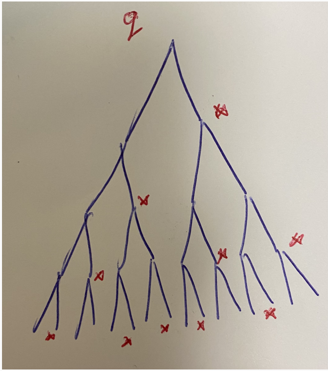


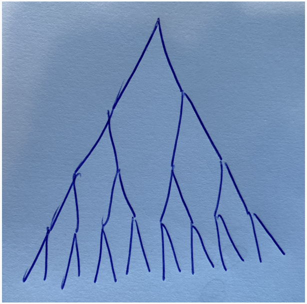


Grigorchuk finished with this question:

Question

Find a formal language description of the portrait of γ (as a language).

Eg: for the element b of his first group, the language would be the set of strings $\{0, 10, 1110, 1^40, 1^60, \dots\}$ which is regular.



Let G be a group, A a finite generating set for G, and H a subset of G.

$$\mathrm{WP}(G,A,H) = \left\{w \in (A \sqcup A^{-1})^* \mid \pi(w) \in H\right\}$$

Eg: if $H = \{1_G\}$, we get the set of words equal to the identity (the usual word problem).

Hartung: if G is given by an L-presentation, and H is a finite index subgroup, then membership in $\mathrm{WP}(G,A,H)$ is decidable.

Mikhailova: $\exists$ finitely generated subgroups of $F_2 \times F_2$ where membership in $\mathrm{WP}(F_2 \times F_2, A, H)$ is undecidable.

An automorphism $g \in \mathrm{Aut}(T_d)$ is

- *finitary* if $g|_v = e$ for all nodes v below some level N_g
- *directed* if $g = \sigma(g|_1, \ldots, \ldots, g|_d)$ and each $g|_i$ is finitary except one, $g|_k$, which is directed
- *bounded* if there is a constant N_g so that $|\{v \in T_d \mid g|_v \neq e, |v| = k\}| \leq N_g$ for all $k \in \mathbb{N}$ (the portrait of g has at most N_g marked nodes at each level).

An automata group is *bounded* if all its elements are bounded.

Sidki: a bounded automata group is generated by finitary and directly elements.

Let G be a bounded automata group with finite generating set A, and

ξ an infinite ray of T_d that is *eventually periodic* ($\xi = pu^\infty$ for some $p, u \in X^*$)

Theorem (Bishop, D'Angeli, Matucci, Nagnibeda, Perego, Rodaro 2024[9], *cf.*[10])

$WP(G, A, Stab(\xi))$ (and its complement) is an ET0L language whose description is effectively constructible from the description of the ray,

and (therefore) membership in it is decidable in quadratic time.

And, since it is a special kind of ET0L language, a nice generating function can be given (to encode the number of words in it of each length).

[9] Bishop, D'Angeli, Matucci, Nagnibeda, Perego, and Rodaro, *On the subgroup membership problem in bounded automata groups*, 2024.

[10] Bishop and Elder, "Bounded automata groups are co-ET0L", 2019.

It follows from their result that the actual word problem for a bounded automata group is the *infinite* intersection of $\mathrm{WP}(G, A, \mathrm{Stab}(\xi))$

(since you can let ξ vary, to be in the intersection you need to fix all of them, so the only element that can do that is e)

Question

Is the word problem for a bounded automata group the intersection of finitely many ET0L languages?

THANKS

REFERENCES

- Bartholdi, Laurent. **"Monadic second-order logic and the domino problem on self-similar graphs".** In: *Groups Geom. Dyn.* 16.4 (2022), pp. 1423–1459. ISSN: 1661-7207,1661-7215. DOI: `10.4171/ggd/689`.
- Bishop, Alex, Daniele D'Angeli, Francesco Matucci, Tatiana Nagnibeda, Davide Perego, and Emanuele Rodaro. ***On the subgroup membership problem in bounded automata groups.*** 2024. arXiv: `2412.08433 [math.GR]`.
- Bishop, Alex and Murray Elder. **"Bounded automata groups are co-ET0L".** In: *Language and automata theory and applications*. Vol. 11417. Lecture Notes in Comput. Sci. Springer, Cham, 2019, pp. 82–94. DOI: `10.1007/978-3-030-13435-8\_6`.
- Bondarenko, Ievgen, Rostislav Grigorchuk, Rostyslav Kravchenko, Yevgen Muntyan, Volodymyr Nekrashevych, Dmytro Savchuk, and Zoran Šunić. **"On classification of groups generated by 3-state automata over a 2-letter alphabet".** In: *Algebra Discrete Math.* 1 (2008), pp. 1–163.
- Davis, Nick, Murray Elder, and Lawrence Reeves. **"Non-contracting groups generated by $(3, 2)$-automata".** In: *Algebra Discrete Math.* 17.1 (2014), pp. 20–32.
- Miller, Alistair and Benjamin Steinberg. ***Homology and K-theory for self-similar actions of groups and groupoids.*** 2024. arXiv: `2409.02359 [math.OA]`.
- Nekrashevych, Volodymyr. ***Self-similar groups.*** Vol. 117. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2005, pp. xii+231. ISBN: 0-8218-3831-8. DOI: `10.1090/surv/117`.
- Ortega, Eduard and Alvaro Sanchez. **"The homology of the groupoid of the self-similar infinite dihedral group".** In: *MATHEMATICA SCANDINAVICA* 128.2 (June 2022). DOI: `10.7146/math.scand.a-129708`.
- Terras, Riho. **"A stopping time problem on the positive integers".** In: *Acta Arith.* 30.3 (1976), pp. 241–252. ISSN: 0065-1036. DOI: `10.4064/aa-30-3-241-252`.