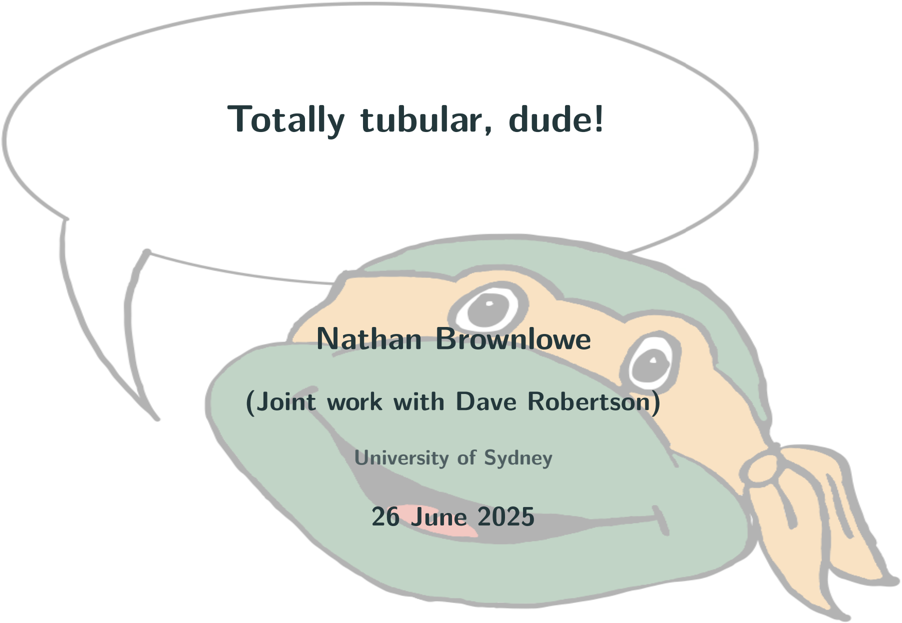

Totally tubular, dude!

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(Joint work with Dave Robertson)

University of Sydney

26 June 2025

Pontryagin duality

Let G be a locally compact abelian group. Then

$$\widehat{G} := \{\text{continuous group homomorphisms } \chi \colon G \to \mathbb{T}\}$$

is a locally compact abelian group under the compact-open topology.

Pontryagin duality: There is an isomorphism $G \to \widehat{\widehat{G}}$ given by

$$g \mapsto (\chi \mapsto \chi(g)).$$

What happens when G is not abelian?

Tannaka–Krein duality for compact groups

Let G be a compact group. Then $\mathrm{Rep}(G)$ is the **category of finite-dimensional continuous complex representations**, where

- objects are pairs (V, π_V), where V is a finite-dimensional complex vector space, and $\pi_V \colon G \to \mathrm{GL}(V)$ is a continuous group homomorphism,
- morphisms are intertwiners: linear maps $T \colon V \to W$ with $\pi_W \circ T = T \circ \pi_V$.

$\mathrm{Rep}(G)$ is a rigid, semisimple tensor category with simple unit.

Tannaka–Krein duality: There is an equivalence $G \mapsto (\mathrm{Rep}(G), F_G)$ of categories between

- the category of compact groups (morphisms are continuous group homomophisms), and
- the category of pairs $(\mathcal{C}, F)$ consisting of rigid, semisimple tensor categories $\mathcal{C}$ with simple unit, and fibre functors $F \colon \mathcal{C} \to \mathrm{Vect}_{\mathbb{C}}^{\mathrm{fd}}$ (morphisms are functors between these categories intertwining the fibre functors).

Compact quantum groups

Let G be a compact group with multiplication $m\colon G \times G \to G$. Then

$$\Delta\colon C(G) \to C(G \times G) \cong C(G) \otimes C(G) \text{ given by } \Delta(f) = f \circ m,$$

satisfies

$$\begin{tikzcd}
C(G) \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowleft"] & C(G) \otimes C(G) \arrow[d, "\mathrm{id} \otimes \Delta"] \\
C(G) \otimes C(G) \arrow[r, "\Delta \otimes \mathrm{id}"'] & C(G) \otimes C(G) \otimes C(G)
\end{tikzcd}$$

and

$$\operatorname{span}\left(\Delta(C(G))(1 \otimes C(G))\right), \operatorname{span}\left(\Delta(C(G))(C(G) \otimes 1)\right) \subseteq C(G) \otimes C(G) \text{ dense.}$$

Compact quantum groups

Definition (Woronowicz 1987)

A **compact quantum group** is a pair (A, Δ) consisting of a unital C^*-algebra A, and a comultiplication $\Delta\colon A \to A \otimes A$ satisfying

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes A \arrow[d, "\mathrm{id} \otimes \Delta"] \\ A \otimes A \arrow[r, "\Delta \otimes \mathrm{id}"'] & A \otimes A \otimes A \end{tikzcd}$$

and

$$\mathrm{span}\,(\Delta(A)(1 \otimes A)),\ \mathrm{span}\,(\Delta(A)(A \otimes 1)) \subseteq A \otimes A \text{ dense.}$$

Example: S_n^+

Theorem (Wang 1998)

Let $A_s(n)$ be the C^*-algebra generated by elements $\{a_{i,j} : 1 \leq i,j \leq n\}$ satisfying

- $a_{i,j} = a_{i,j}^* = a_{i,j}^2$,
- $\sum_{i=1}^{n} a_{i,j} = 1 = \sum_{j=1}^{n} a_{i,j}$.

Then there is a comultiplication $\Delta\colon A_s(n) \to A_s(n) \otimes A_s(n)$ satisfying

$$\Delta(a_{i,j}) = \sum_{k=1}^{n} a_{i,k} \otimes a_{k,j},$$

and the pair $S_n^+ = (A_s(n), \Delta)$ is a compact quantum group.

This is "quantum S_n" because if we abelianise S_n^+, then each $a_{i,j}$ becomes the coordinate function $f_{i,j} \in C(S_n)$ given by $f_{i,j}(\sigma) = \sigma_{i,j}$ for $\sigma \in S_n$.

Representations of CQGs

Let (A, Δ) be a compact quantum group.

A **finite-dimensional representation** u of A is a finite-dimensional Hilbert space $\mathcal{H}_u$ (with dimension $d_u := \dim \mathcal{H}_u$) and an invertible element $u \in \mathcal{B}(\mathcal{H}_u) \otimes A \cong M_{d_u}(A)$ satisfying

$$\Delta(u_{i,j}) = \sum_{k=1}^{d_u} u_{i,k} \otimes u_{k,j}.$$

for all $1 \leq i, j \leq d_u$.

Tannaka–Krein for CQGs

Let $G = (A, \Delta)$ be a compact quantum group.

We write $\mathrm{Rep}(G)$ for the category of finite-dimensional unitary representations of G, where morphisms are again intertwiners. $\mathrm{Rep}(G)$ is a rigid C^*-tensor category with simple unit.

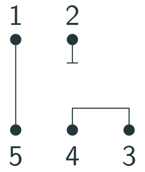
We write $F_G \colon \mathrm{Rep}(G) \to \mathrm{Hilb}^{\mathrm{fd}}$ for the forgetful functor.

Tannaka–Krein for CQGs (Woronowicz 1988): There is an equivalence $G \mapsto (\mathrm{Rep}(G), F_G)$ between the categories of compact quantum groups and pairs $(\mathcal{C}, F)$ consisting of rigid C^*-tensor categories with simple unit, and faithful unitary fibre functors $F \colon \mathcal{C} \to \mathrm{Hilb}^{\mathrm{fd}}$.

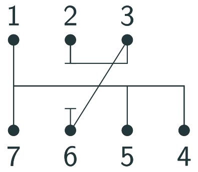
Partition categories

A **partition** $p \in P(m,n)$ is a decomposition of $m+n$ points into blocks. For example

$p = \{\{1,5\},\{2\},\{3,4\}\} \in P(2,3)$ is

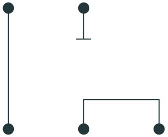


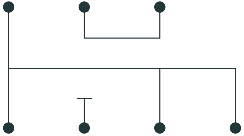
$q = \{\{1,4,5,7\},\{2\},\{3\},\{6\}\} \in P(3,4)$ is

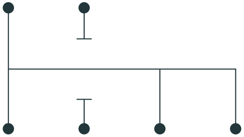


Partition categories

We can compose partitions

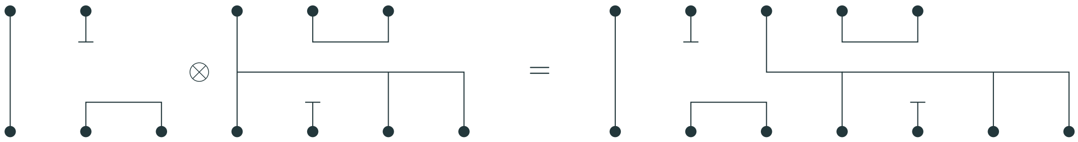
$p =$ 

$q =$ 

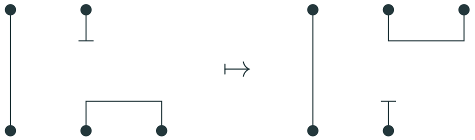
$\mapsto$  $= q \circ p$

Partition categories

We can take tensor products of partitions



and their adjoints



Partition categories

A **partition category** $\mathcal{P}$ is a collection $\mathcal{P} = (\mathcal{P}(m,n))_{m,n\in\mathbb{N}}$, with $\mathcal{P}(m,n) \subseteq P(m,n)$, and such that

- $\mathcal{P}$ is closed under composition, tensor products, and adjoints,
- $\in \mathcal{P}(1,1)$, and
-  $\in \mathcal{P}(0,2)$.

The objects of a partition category $\mathcal{P}$ is $\mathbb{N}$.

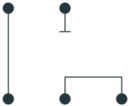
Examples include the category of noncrossing partitions NC, and the category of all pair partitions $\mathcal{P}_2$, where every block in a partition contains exactly two elements.

Linear maps associated to partition diagrams

Let X be a nonempty finite set. For each $n \in \mathbb{N}$ let X^n be the set of words in X, and let $\mathcal{H}_n := \mathbb{C}^{X^n}$ with canonical basis $\{e_x : x \in X^n\}$.

For a partition $p \in \mathcal{P}(m, n)$ label the top and bottom points (both left to right) with $x \in X^m$ and $y \in X^n$ respectively. Define $\delta_p(x, y)$ to be 1 if all points in a given block in p carry the same label, and 0 otherwise.

Example. For $X = \{a, b\}$

$p =$ 

$$\delta_p(ab, abb) = 1, \quad \delta_p(ab, bbb) = 0$$

Then we define $T_p \colon \mathbb{C}^{X^m} \to \mathbb{C}^{X^n}$ by $T_p(e_x) = \sum_{\delta_p(x,y)=1} e_y$.

So for example we have $T_p(e_{ab}) = e_{abb} + e_{aaa}$.

Easy quantum groups

A **compact matrix quantum group** (A, Δ) is one in which there exists $n \in \mathbb{N}$ and unitary $u \in M_n(A)$ such that

- $\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kl}$, and
- the entries $\{u_{ij}\}$ generate a dense $*$-subalgebra of A.

Definition (Banica–Speicher 2009)

A compact matrix quantum group is called an **easy quantum group** if there is a category of partitions $\mathcal{P}$ such that

$$\mathrm{Mor}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}\{T_p : p \in \mathcal{P}(k, l)\}.$$

If we take

- $\mathcal{P} = P$, we get S_n
- $\mathcal{P} = \mathrm{NC}$, we get S_n^+
- $\mathcal{P} = \mathcal{P}_2$, we get O_n
- $\mathcal{P} = \mathcal{P}_2 \cap \mathrm{NC}$, we get O_n^+

Example: $\mathbb{A}_X$

Let X be a finite nonempty set. What is the quantum automorphism group of an infinite homogeneous rooted tree X^*?

Let $\mathbb{A}_X$ be the universal C^*-algebra generated by elements $a_{u,v}$, $u, v \in X^n$, satisfying

- $a_{\emptyset,\emptyset} = 1$
- $a_{u,v} = a_{u,v}^* = a_{u,v}^2$
- $a_{u,v} = \sum_{y \in X} a_{ux,vy} = \sum_{z \in X} a_{uz,vx}$.

Theorem (B–Robertson 2025)

The C^*-algebra $\mathbb{A}_X$ is a compact quantum group with comultiplication $\Delta \colon \mathbb{A}_X \to \mathbb{A}_X \otimes \mathbb{A}_X$ satisfying $\Delta(a_{u,v}) = \sum_{|w|=|u|=|v|} a_{u,w} \otimes a_{w,v}$.

If we abelianise, $\mathbb{A}_X$ becomes $C(\mathrm{Aut}(X^*))$, where $\mathrm{Aut}(X^*)$ is the automorphism group of the infinite homogeneous rooted tree X^*, and each $a_{u,v}$ becomes the characteristic function on $\{g \in \mathrm{Aut}(X^*) : g(v) = u\}$.

Question

What can we say about the category associated to $(\mathbb{A}_X, \Delta)$ via Tannaka–Krein duality?

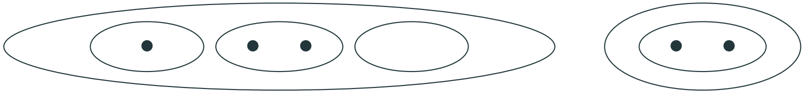
Tubular partitions

Define $\mathcal{C}_1 := \mathrm{NC}$.

Suppose $\mathcal{C}_k$ is given, and define

$$\mathcal{C}_{k+1} := \{(m; \alpha_1, \ldots, \alpha_m) : m \in \mathcal{C}_1,\ \alpha_i \in \mathcal{C}_k\}.$$

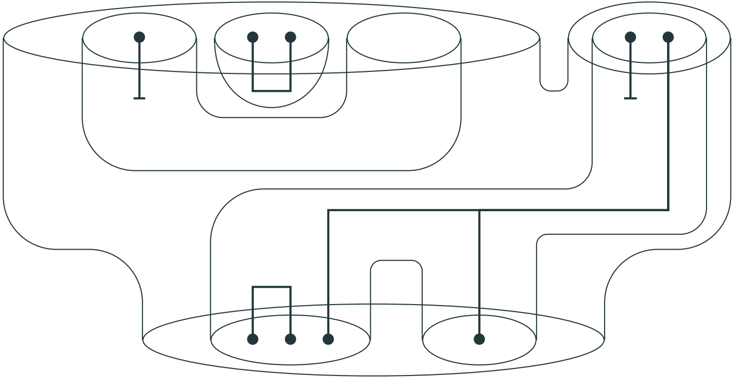
We represent these objects with dots and ellipses



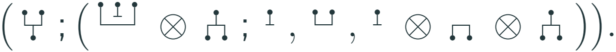
$(2; (3; 1, 2, 0), (1; 2)) \in \mathcal{C}_3$

Tubular partitions

Morphisms are tubes!



This is the morphism

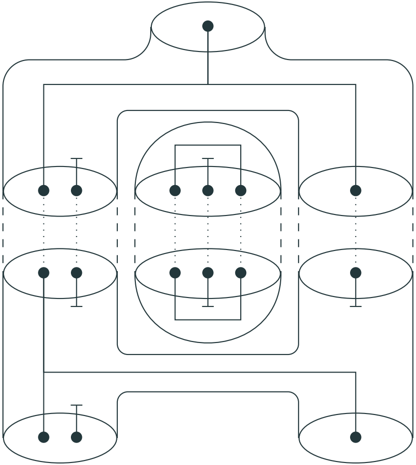


Tubular partitions

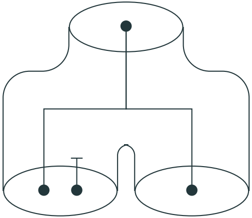
We compose, tensor, and take adjoints as we did with partitions. For instance

$\pi =$

$\rho =$



$\mapsto$



$= \rho \circ \pi$

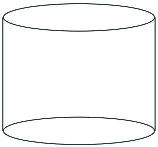
Tubular partitions

We want to identify objects like

with

and morphisms like

with



We do this formally with inclusions $I_k \colon \mathcal{C}_k \hookrightarrow \mathcal{C}_{k+1}$.

Definition (B-Robertson)

The category of tubular partitions $\mathcal{T}$ is the union $\mathcal{T} = \bigcup_{k=1}^{\infty} \mathcal{C}_k$, where we identify $\mathcal{C}_k$ with its image $I_k(\mathcal{C}_k) \subset \mathcal{C}_{k+1}$.

Main result

To each $\alpha \in \mathcal{C}_k$ we define a set of generalised words X^α, and associate a Hilbert space $\mathcal{H}_\alpha := \mathbb{C}^{X^\alpha}$. To each morphism $\rho \in \mathcal{C}_k(\alpha, \beta)$ we define an intertwiner $T_\rho \colon \mathcal{H}_\alpha \to \mathcal{H}_\beta$ analogous to those associated to partitions.

Theorem (B-Robertson)

Let X be a nonempty finite set. Let $\mathcal{T}$ be the category of tubular partitions, and let $\mathcal{R}_X$ be the smallest rigid C^*-tensor category containing the Banach spaces

$$\mathcal{R}_X(\alpha, \beta) := \mathrm{span}\{T_\rho : \rho \in \mathcal{T}(\alpha, \beta)\} \subseteq B(\mathcal{H}_\alpha, \mathcal{H}_\beta),$$

where $\alpha, \beta \in \mathcal{T}$. Then the compact quantum group associated to $\mathcal{R}_X$ via Tannaka-Krein duality is $(\mathbb{A}_X, \Delta)$.

Key structural tool

Classically, there is a function $\mathrm{Aut}(X^*) \times X \to X \times \mathrm{Aut}(X^*)$ given by

$$(g, x) \mapsto (g \cdot x, g|_x).$$

The quantum analogue is a homomorphism $\psi : \mathbb{C}^X \otimes \mathbb{A}_X \to \mathbb{A}_X \otimes \mathbb{C}^X$.

Given $u = \mathrm{Rep}(\mathbb{A}_X)$ on $\mathcal{H}$, we use ψ to get $\psi(u) = \mathrm{Rep}(\mathbb{A}_X)$ on $\mathbb{C}^X \otimes \mathcal{H}$ via

$$\psi(e_x \otimes u_{ij}) = \sum_{y \in X} \psi(u)_{(x,i),(y,j)} \otimes e_y.$$

Key structural tool

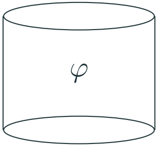
We see this on the tubes via the endofunctor $\Psi \in \mathrm{End}(\mathcal{T})$ given by

$$\Psi(\alpha) = (1; \alpha) \quad \text{and} \quad \Psi(\varphi) = (\,|\,; \varphi),$$

for an object α and morphism φ.

In pictures

$\Psi \colon \alpha \mapsto$ 

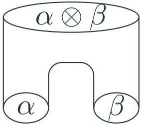
$\Psi \colon \varphi \mapsto$ 

Key structural tool

For instance, $\mathcal{C}_{k+1}$ is generated by

$$\{\Psi(\varphi) : \varphi \in \mathrm{Hom}(\mathcal{C}_k)\} \cup \{P_{\alpha,\beta} : \alpha, \beta \in \mathcal{C}_k\} \cup \{\ \},$$

where

$P_{\alpha,\beta}$ is the pants 

and

is really the dome  in $\mathcal{C}_{k+1}$.

Future questions

- What can we say about the category associated to a general self-similar compact quantum group?
- Do they involve $\mathcal{T}$ and Ψ?
- Can we define a self-similar C^*-tensor category?
- Will anyone care?

THANKS!